IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Differential calculus and gauge theory on finite sets

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1994 J. Phys. A: Math. Gen. 27 3159

(http://iopscience.iop.org/0305-4470/27/9/028)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 01/06/2010 at 23:58

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 27 (1994) 3159–3178. Printed in the UK

Differential calculus and gauge theory on finite sets*

Aristophanes Dimakis† and Folkert Müller-Hoissen‡

† Department of Mathematics, University of Crete, GR-71409 Iraklion, Greece
‡ Institut für Theoretische Physik, Bunsenstrasse 9, D-37073 Göttingen, Germany

Received 9 August 1993

Abstract. We develop differential calculus and gauge theory on a finite set G. An elegant formulation is obtained when G is supplied with a group structure and in particular for a cyclic group. Connes' two-point model (which is an essential ingredient of his reformulation of the standard model of elementary particle physics) is recovered in our approach. Reductions of the universal differential calculus to 'lower-dimensional' differential calculi are considered. The 'complete reduction' leads to a differential calculus on a periodic lattice.

1. Introduction

Again and again over the years, arguments have been given to assign a more fundamental role to discrete spaces rather than to the continuum and attempts were made to develop corresponding physical theories (see [1] for some early examples). Such an idea has been pursued by Finkelstein since 1968 [2] culminating in a forthcoming book. Classical and quantum field theory on discrete spaces has been considered, in particular, in [3]. The finiteness of the entropy of a black hole (and the corresponding finiteness of the number of bits of information that can be stored there) led 't Hooft to speculate about a discrete (cellular automaton) structure of spacetime at the Planck scale [4]. Further interesting ideas about discreteness of space and time can be found in [5, 6], for example.

More recently, concepts of differential geometry were extended to discrete spaces (and even non-commutative algebras). In the framework of non-commutative geometry, finite spaces have been considered to build models of elementary particle physics [7] (see also [8]). The present work provides a general approach to the differential geometry of such spaces. It has been inspired by recent papers of Sitarz [9] who treated the case of discrete groups (see also [10]).

We take the point of view that some form of differential calculus is the very basic structure necessary to formulate physical models and, in particular, dynamics of fields on some space. Our belief in the physical relevance of this mathematical structure is partly based on the observation made in [11, 12] that lattice theories (in particular their Lagrangian and action) are obtained from continuum theories in a universal way simply by a certain deformation of the ordinary calculus of differential forms. In this case functions and differentials satisfy non-trivial commutation relations depending on the lattice spacings (for vanishing lattice spacings they commute and one recovers the ordinary differential calculus). Another deformation of the ordinary differential calculus was shown to be related with stochastic calculus [13] and 'proper time' formulations [14] of physical theories [15].

* Work supported by the Heraeus-Foundation.

 © 1994 IOP Publishing Ltd

In section 2 we introduce the universal differential calculus (*universal differential envelope* [10, 16]) on an arbitrary finite set of N elements. Section 3 shows how to formulate gauge theory on a finite set. Of special interest is the case when the set is supplied with a group structure. This is the subject of section 4. In section 5, we consider the group structure $\mathbb{Z}_N$ in detail.

In the gauge theory formalism based on the universal differential calculus a connection (gauge potential) on a finite set can provide (a set of) Higgs fields [7]. This observation stimulated the use of non-commutative geometry for model building in particle physics [7, 8]. We briefly discuss it for the two-point space in section 5.2.

The universal differential calculus on a finite set of order N associates with it $N-1$ linearly independent differentials. On the other hand, we know that there are 'smaller' differential calculi. In particular, it is possible to have the N points 'on a closed line', i.e. embedded as a lattice in one dimension. This configuration is described by a single coordinate y which satisfies the commutation relation

$$y\,\mathrm{d}y = q\,\mathrm{d}y\,y$$

with its differential $\mathrm{d}y$ where q is an Nth primitive root of unity (see [11]). There is a natural way from the universal differential calculus to this 'reduced' differential calculus. Related with the fact that one always has the group structure $\mathbb{Z}_N$ on a set of N elements, there is a function y such that $y^N = 1$. Expressing the universal differential calculus in terms of the functions y^n, $n = 0, \ldots, N-1$, we can consistently add the above relation so that the $(N-1)$-dimensional universal calculus is reduced to a one-dimensional differential calculus. Details are presented in section 6. The reduced differential calculus (and its higher-dimensional generalization) gives a convenient universal framework to formulate and describe physical models on a (closed) lattice [11, 12].

Between the universal differential calculus (which assigns an $(N-1)$-dimensional polyhedron to a set of N points) and the one-dimensional (periodic lattice) calculus there are other differential calculi. It is our concern what kind of geometric† structures can be associated with them. Section 6 explores some of the possibilities.

A reduction of the universal differential calculus induces a corresponding reduction of structures built on it, like gauge theory. In this way one can approach field theory on finite sets.

Section 7 contains some conclusions.

2. Differential calculus on a finite set

Let G be a finite set with N elements and $\mathcal{A}$ the algebra of $\mathbb{C}$-valued functions on G with the usual pointwise multiplication of functions,

$$(ff')(g) = f(g)\,f'(g) \qquad \forall f, f' \in \mathcal{A},\ g \in G\,. \tag{2.1}$$

$\mathcal{A}$ is a commutative, associative and unital complex algebra. With each $g \in G$ we associate a function $x_g \in \mathcal{A}$ such that

$$x_g(g') = \delta_{g,g'} \qquad \forall g' \in G\,. \tag{2.2}$$

The functions x_g satisfy the identities

$$\sum_{g\in G} x_g = \mathbf{1} \qquad \text{(the unit in } \mathcal{A}) \tag{2.3}$$

† The notion 'geometric' here refers to a connection structure on the set of points in the sense of graphs. We do not consider a metric (or distance function) on the point set in this work.

$$x_g\, x_{g'} = \delta_{gg'}\, x_g \tag{2.4}$$

which will be frequently used in the following. As a consequence of these two identities, every function $f \in \mathcal{A}$ has an expansion

$$f = \sum_g f(g)\, x_g \tag{2.5}$$

which shows that the functions x_g span $\mathcal{A}$ linearly over $\mathbb{C}$. Furthermore,

$$x_g\, f = x_g\, f(g) \qquad \forall f \in \mathcal{A}\,. \tag{2.6}$$

The complex conjugate of $f \in \mathcal{A}$ will be denoted as f^*.

We extend $\mathcal{A} =: \Omega^0$ to a differential algebra via the action of an *exterior derivative* operator d. It maps elements of $\mathcal{A}$ into (formal) differentials which span the space Ω^1 of 1-forms as an $\mathcal{A}$-bimodule, and furthermore r-forms into $(r+1)$-forms, i.e. $d\,:\,\Omega^r \to \Omega^{r+1}$. We require

$$\mathrm{d}1 = 0 \tag{2.7}$$

$$\mathrm{d}^2 = 0 \tag{2.8}$$

$$\mathrm{d}(\omega\omega') = \mathrm{d}\omega\, \omega' + (-1)^r\, \omega\, \mathrm{d}\omega' \tag{2.9}$$

where ω and ω' are r- and r'-forms, respectively. Ω denotes the space $\bigoplus_{r=0}^{\infty} \Omega^r$ of all forms.

Now (2.3) implies

$$\sum_g \mathrm{d}x_g = 0 \tag{2.10}$$

(2.5) leads to

$$\mathrm{d}f = \sum_g f(g)\, \mathrm{d}x_g \tag{2.11}$$

since $f(g)$ are constants, and (2.6) yields

$$f\, \mathrm{d}x_g = f(g)\, \mathrm{d}x_g - \mathrm{d}f\, x_g\,. \tag{2.12}$$

Acting with d on these relations does not lead to further relations. Equation (2.12) has the form of commutation relations between differentials and elements of $\mathcal{A}$. It should be noticed that these relations are simply a consequence of the Leibniz rule (2.9).

We will now choose an element $e \in G$ once and for all†. Equation (2.10) shows that the differentials $\mathrm{d}x_g$ are linearly dependent. The differentials $\mathrm{d}x_g$ with $g \in G \setminus \{e\} =: G'$ are linearly independent, however. Instead of $\sum_{g\in G'}$ we will write $\sum'_g$ in the following. Then

$$\mathrm{d}f = \sum_g{}' \mathrm{d}x_g\, [f(g) - f(e)] \tag{2.13}$$

which shows that $\mathrm{d}f = 0$ iff f takes the same value at each element of G.

The x_g are real functions, i.e.

$$(x_g)^* = x_g\,. \tag{2.14}$$

The complex conjugation on $\mathcal{A}$ can be extended to an involution of the differential algebra by [7]

$$(f_1\, \mathrm{d}f_2 \cdots \mathrm{d}f_k)^* = \mathrm{d}(f_k^*) \cdots \mathrm{d}(f_2^*)\, f_1^* \tag{2.15}$$

† In the case of a group a natural choice will be the unit element. In general, however, there will not be a distinguished choice of e.

where $f_\ell \in \mathcal{A}$.

If we put an ordering on the elements of G, i.e. $g_0, \ldots, g_{N-1}$, then there is another natural choice of an involution on $\mathcal{A}$. It is determined by

$$(x_i)^\star = x_{N-i} \tag{2.16}$$

where $x_i := x_{g_i}$. Again, it extends to $\Omega(\mathcal{A})$ via

$$(f_1\, \mathrm{d}f_2 \cdots \mathrm{d}f_k)^\star = \mathrm{d}(f_k^\star) \cdots \mathrm{d}(f_2^\star)\, f_1^\star\,. \tag{2.17}$$

On $\mathbb{C}$, the involution should coincide with complex conjugation. We will find the $\star$-involution of importance when we consider reductions of the differential calculus in section 6.

Let us introduce the 1-forms

$$\theta_{ij} = \mathrm{d}x_i\, x_j \tag{2.18}$$

for $i \neq j$. It follows that

$$\theta_{jk}\, x_i = \theta_{jk}\, \delta_{ki} \qquad x_i\, \theta_{jk} = \delta_{ij}\, \theta_{jk} \qquad j \neq k \tag{2.19}$$

so that the θ_{jk} are common eigen-1-forms for all x_i. As a consequence, if we impose the condition $\theta_{jk} = 0$ for fixed j and k on the differential calculus, then we do not obtain further relations by multiplication with elements of $\mathcal{A}$. We shall return to this observation in section 6. The θ_{ij} are a basis of the space of 1-forms as a complex vector space. In particular, we have

$$\mathrm{d}f = \sum_{i \neq j} \theta_{ij}\, [f(g_i) - f(g_j)] = \Big[-\sum_{i\neq j} \theta_{ij},\, f \Big] \tag{2.20}$$

for $f \in \mathcal{A}$. The two involutions introduced above act on θ_{ij} in the following way:

$$\theta_{ij}^* = -\theta_{ji} \tag{2.21}$$

$$\theta_{ij}^\star = -\theta_{N-j,N-i}\,. \tag{2.22}$$

The simple relations which the x_i and the θ_{jk} satisfy suggest the following construction of finite-dimensional matrix representations of the differential calculus. Let E_{ij} denote the $(N \times N)$ matrix with components $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. The matrices

$$\rho(x_i) = \begin{pmatrix} E_{ii} & 0 \\ 0 & E_{ii} \end{pmatrix} \qquad \rho(\theta_{ij}) = \begin{pmatrix} 0 & C_{ij}\, E_{ij} \\ C'_{ij}\, E_{ij} & 0 \end{pmatrix} \tag{2.23}$$

with (non-zero) constants C_{ij}, C'_{ij} yield a representation of the differential algebra. However, only functions and 1-forms are *faithfully* represented. If $C_{ij}^* = -C'_{ji}$, our first involution acts by Hermitian conjugation on the matrices. To represent the second involution, an additional 'reflection' of the indices has to be performed. The 'doubling' of the matrices in (2.23) is necessary for $N > 2$ to account for the $\mathbb{Z}_2$-grading of the differential algebra. The case $N = 2$ is special in this respect (see also appendices A and B). More general representations are given by

$$\rho(x_i) = \begin{pmatrix} \mathbf{1}_M \otimes E_{ii} & 0 \\ 0 & \mathbf{1}_M \otimes E_{ii} \end{pmatrix} \qquad \rho(\theta_{ij}) = \begin{pmatrix} 0 & C_{ij} \otimes E_{ij} \\ C'_{ij} \otimes E_{ij} & 0 \end{pmatrix} \tag{2.24}$$

where C_{ij} and C'_{ij} are now $(M \times M)$-matrices and $\mathbf{1}_M$ denotes the $(M \times M)$ unit matrix.

3. Gauge fields on a finite set

Using the differential calculus introduced in the previous section, gauge theory can now be formulated on a finite set G. Section 3.1 deals with the concept of a connection and its field strength (curvature). In section 3.2 we consider the covariant derivative of a field which transforms according to a representation of the gauge group. If a conjugation is given on the space of fields such that it is compatible with the connection, then the connection turns out to be anti-Hermitian for a unitary gauge group, in analogy with the continuum case.

3.1. Connection and field strength

Let

$$A = \sum_g \mathrm{d}x_g \, A_g \tag{3.1}$$

be a *connection 1-form* which transforms under a gauge transformation according to the familiar rule

$$A' = U \, A \, U^{-1} - \mathrm{d}U \, U^{-1} \tag{3.2}$$

where U is a function on G with values in a matrix group. Inserting (3.1) in (3.2) and using (2.12) we find

$$\mathrm{d}U \, (1 + \sum_g x_g \, A_g) = \sum_g \mathrm{d}x_g \, [U(g) \, A_g - A'_g \, U] \, . \tag{3.3}$$

This equation is satisfied when†

$$\sum_g x_g \, A_g = -1 \tag{3.4}$$

$$A'_g = U(g) \, A_g \, U^{-1} \, . \tag{3.5}$$

One has to check that (3.4) is gauge-invariant. Indeed,

$$\sum_g x_g \, A'_g = \sum_g x_g \, U(g) \, A_g \, U^{-1} = U \sum_g x_g \, A_g \, U^{-1} = -U \, U^{-1} = -1 \, . \tag{3.6}$$

Because of (2.10) the coefficients A_g in (3.1) are not uniquely determined by the left-hand side of (3.1). The corresponding freedom is fixed, however, by the condition (3.4).

Multiplying (3.4) by $x_{g'}$ and using (2.6) yields

$$x_g \, (A_g(g) + 1) = 0 \tag{3.7}$$

so that

$$A_g(g) = -1 \qquad \forall g \in G \, . \tag{3.8}$$

From (3.2) one finds that

$$F = dA + A \, A = \sum_{g,g'} \mathrm{d}x_g \, \mathrm{d}x_{g'} \, A_g(g') \, A_{g'} \tag{3.9}$$

transforms according to $F \mapsto U \, F \, U^{-1}$.

† The last equation may admit other solutions. This will not be discussed further in the present work. Note that the 1 appearing in (3.3) and on the right-hand sides of (3.4) and (3.6) has to be understood as **1** times the unit matrix of the gauge group.

3.2. Covariant derivatives

Using (3.2) we have

$$U\,\mathrm{d}x_g = \mathrm{d}x_g\,U(g) - \mathrm{d}U\,x_g = \mathrm{d}x_g\,U(g) - U\,A\,x_g + A'\,x_g\,U(g) \tag{3.10}$$

which shows that

$$Dx_g := \mathrm{d}x_g + A\,x_g \tag{3.11}$$

transforms covariantly,

$$D'x_g = U\,Dx_g\,U(g)^{-1}\,. \tag{3.12}$$

Furthermore, it satisfies

$$\sum_g Dx_g = A\,. \tag{3.13}$$

Let us consider a field ψ on G as an element of a module $\mathcal{V} := \mathcal{A}^n$. If the gauge group acts on it according to $\psi \mapsto U\,\psi$, then

$$D\psi := \mathrm{d}\psi + A\,\psi \tag{3.14}$$

has the same transformation property as a consequence of (3.2). Using (2.5) one finds

$$D\psi = \sum_g Dx_g\,\psi(g)\,. \tag{3.15}$$

From (3.13) and (3.4) one obtains the identity

$$\sum_g Dx_g\,A_g = 0\,. \tag{3.16}$$

Furthermore, using the Leibniz rule for d, (2.6), (3.8) and (2.3), one can express $D\psi$ as a right-form,

$$D\psi = \sum_g{}' \overleftarrow{\nabla}_g\,\psi\,\mathrm{d}x_g \tag{3.17}$$

with

$$\overleftarrow{\nabla}_g\,\psi = \sum_{g'} x_{g'}\,[A_{g'}(e)\,\psi(e) - A_{g'}(g)\,\psi(g)]\,. \tag{3.18}$$

Let a conjugation be given which maps $\psi \in \mathcal{V}$ to an element $\psi^\dagger$ of the dual module $\mathcal{V}^*$ such that $(\omega\psi)^\dagger = \psi^\dagger\omega^*$ (or with ω^* replaced by $\omega^\star$). If U is unitary, i.e. $U^\dagger = U^{-1}$, then the assumption

$$(D\psi)^\dagger = D(\psi^\dagger) \tag{3.19}$$

implies

$$\mathrm{d}\psi^\dagger + \psi^\dagger\,A^\dagger = (\mathrm{d}\psi + A\,\psi)^\dagger = \mathrm{d}\psi^\dagger - \psi^\dagger\,A \tag{3.20}$$

and therefore

$$A^\dagger = -A\,. \tag{3.21}$$

4. Differential calculus and gauge theory on a finite group

Differential geometric structures on Lie groups like Maurer–Cartan forms play an important role in the construction of physical models and in particular in the formulation of gauge theories as structures on principal fibre bundles. It is therefore of interest that these structures can also be formulated on discrete groups (see also [9]).

Since, on a finite set G it is always possible to introduce a group structure, such a structure can be used to rewrite the differential calculus and gauge theory introduced in the previous sections in a different and often simpler form. This will be shown in the following two subsections.

4.1. Differential calculus on a finite group

Let us consider a group structure on a finite set G with group multiplication $(g, g') \mapsto gg'$. For the element $e \in G$ we choose the unit element of the group. Right and left actions on the algebra $\mathcal{A}$ of functions on G are then given by

$$(R_g f)(g') := f(g'g) \qquad (L_g f)(g') := f(gg') . \tag{4.1}$$

One finds that the 1-forms

$$\theta_g := \sum_{g'} \mathrm{d}x_{g'g}\, x_{g'} = \sum_{g'} \mathrm{d}x_{g'}\, x_{g'g^{-1}} \tag{4.2}$$

are left-invariant, i.e.

$$L_{g''}\theta_g := \sum_{g'} \mathrm{d}x_{g''^{-1}g'g}\, x_{g''^{-1}g'} = \theta_g . \tag{4.3}$$

The 1-forms θ_g are, in this sense, analogues of left-invariant Maurer–Cartan forms on a Lie group. They satisfy the identity

$$\sum_g \theta_g = 0 \tag{4.4}$$

as a consequence of (2.10). Furthermore,

$$\mathrm{d}x_g = \sum_{g'} \theta_{g'}\, x_{gg'^{-1}} . \tag{4.5}$$

Using (4.2), (2.12), (2.6) and (2.3), one derives the relation

$$f\,\theta_g = \theta_g\, R_g f - \delta_{g,e}\, \mathrm{d}f . \tag{4.6}$$

As a consequence of (4.6)

$$\mathrm{d}f = [\theta_e\, ,\, f] \tag{4.7}$$

which assigns a particular role to θ_e. Furthermore, we obtain

$$\mathrm{d}f = \sum_g \theta_g\, R_g f = \sum_g{}' \theta_g\,(R_g - 1) f \tag{4.8}$$

using (2.5), (4.5), (2.6), (2.3) and (4.4). Acting with d on (4.2) leads to

$$\mathrm{d}\theta_g = \sum_{g'} \theta_{g'}\, \theta_{gg'^{-1}} \tag{4.9}$$

which resembles Maurer–Cartan equations. The expression obtained by acting with d on (4.8), using (4.9) and (4.8), has to vanish as a consequence of $\mathrm{d}^2 = 0$. It vanishes identically, however, so that we do not obtain relations between 1-forms.

Using (4.2), (2.14), (2.15), (2.9), (2.6) and (2.10), one finds

$$(\theta_g)^* = -\theta_{g^{-1}} . \tag{4.10}$$

In case of the ⋆-involution (cf section 2) one obtains

$$(\theta_g)^\star = -\theta_g . \tag{4.11}$$

It is also possible, of course, to introduce *right*-invariant Maurer–Cartan forms. All the above relations for θ_g have corresponding counterparts.

If the number N of elements of G is a prime number, then the only possible group structure is $\mathbb{Z}_N$ [22]. In the general case, each group with N elements must be a subgroup of the symmetric (permutation) group S_N. For a finite set, the symmetric group is of particular interest since it plays the role of the homeomorphism (or diffeomorphism) group of topological spaces or manifolds. The symmetric group and its representations are therefore expected to be important in an approach towards a theory of gravity on a finite discrete spacetime.

4.2. Gauge fields on a finite group

Let us write the connection 1-form A in terms of the 1-forms θ_g,

$$A = \sum_g \theta_g \, P_g . \tag{4.12}$$

Then

$$A_{g'} = \sum_g x_{g'g^{-1}} \, P_g \tag{4.13}$$

and the condition (3.4) translates into the much simpler equation

$$P_e = -1 \tag{4.14}$$

(where the 1 stands for **1** times the unit matrix of the gauge group). Under a gauge transformation,

$$P'_g = (R_g U) \, P_g \, U^{-1} . \tag{4.15}$$

Using (4.6) and (3.2), the transformation of the 1-forms

$$\tilde\theta_g := \theta_g + \delta_{g,e} \, A \tag{4.16}$$

is found to be given by

$$U \, \tilde\theta_g = \tilde\theta'_g \, R_g U . \tag{4.17}$$

They satisfy

$$\sum_g \tilde\theta_g = A . \tag{4.18}$$

The field strength of A is

$$F = \sum_{g,g'} \theta_g \, \theta_{g'} \, [P_{gg'} + (R_{g'} P_g) \, P_{g'}] = \sum_{g,g'} \tilde\theta_g \, \tilde\theta_{g'} \, [P_{gg'} + (R_{g'} P_g) \, P_{g'}] . \tag{4.19}$$

For the covariant derivative of ψ we obtain the expression

$$D\psi = \sum_g \tilde\theta_g \, [R_g \psi + P_g \psi] . \tag{4.20}$$

With the $*$-involution, using (4.12) and (4.10), the condition (3.21) translates into

$$P_g^\dagger = R_g P_{g^{-1}} \qquad (\forall g \in G) . \tag{4.21}$$

Using the $\star$-involution instead, we obtain

$$P_g^\dagger = R_{g^{-1}} P_g \qquad (\forall g \in G) . \tag{4.22}$$

Remark. Gauge theory on finite groups has been discussed in previous work by Sitarz [9]. His results are not quite in accordance with ours. Moreover, our approach stresses the fact that a group structure is just an auxiliary structure which can be used to deal with the differential calculus and gauge theory on G in a more convenient way.

5. The case $G = \mathbb{Z}_N$

In this section we study the differential calculus on a finite set G of N elements with the help of the group structure of $\mathbb{Z}_N$. This leads to another look at the matrix representation of the differential calculus in section 5.1. The example of gauge theory on $\mathbb{Z}_2$ is discussed in section 5.2 making contact with Connes' two-point space model [7].

If we describe $\mathbb{Z}_N$ as the set of numbers $\{0, 1, \ldots, N-1\}$ with addition modulo N as the group structure, then the functions x_m, $m = 0, \ldots, N-1$ defined by

$$x_m(n) = \delta_{m,n} \tag{5.1}$$

correspond to the functions x_g of section 2. Let us introduce a new function

$$y := \sum_{n=0}^{N-1} q^n x_n \tag{5.2}$$

with $q \in \mathbb{C}$ a primitive Nth root of unity, i.e. $q^N = 1$. Then

$$y^n = \sum_{m=0}^{N-1} q^{mn} x_m \tag{5.3}$$

and, in particular, $y^N = \mathbf{1}$. Note that the last equation describes the N-point set in the simplest possible algebraic way. It replaces the set of equations (2.3) and (2.4).

Like the x_n, $n = 0, \ldots, N-1$, the set of functions $y^0, \ldots, y^{N-1}$ also span the algebra $\mathcal{A}$ of functions on $\mathbb{Z}_N$. Using the identity

$$\sum_{m=0}^{N-1} q^{nm} = N\, \delta_{n,0} \tag{5.4}$$

(5.3) can be inverted,

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} q^{-mn}\, y^m . \tag{5.5}$$

A function f on G can be written as

$$f = \sum_{n=0}^{N-1} y^n f_n \tag{5.6}$$

where

$$f_n = \frac{1}{N} \sum_{m=0}^{N-1} q^{-mn} f(m) . \tag{5.7}$$

The two involutions introduced in section 2 act on y as follows:

$$y^* = y^{-1} \qquad y^\star = y\,. \tag{5.8}$$

We shall now rewrite the differential calculus introduced in section 2 in terms of y^n. Equation (2.12) implies

$$y\,\mathrm{d}y^n + \mathrm{d}y\, y^n = \mathrm{d}y^{n+1} \qquad (n = 1, \ldots, N-2) \tag{5.9}$$

$$y\,\mathrm{d}y^{N-1} + \mathrm{d}y\, y^{N-1} = 0\,. \tag{5.10}$$

The 1-forms θ_g introduced in section 4 now take the form

$$\theta_n = \frac{1}{N} \sum_{m=1}^{N-1} q^{-mn}\, \mathrm{d}y^m\, y^{-m}\,. \tag{5.11}$$

The right action on $\mathcal{A}$ is given by $(R_n f)(y) = f(q^n y)$. Equation (4.6) then implies†

$$y\,\theta_n = q^n\, \theta_n\, y \qquad (n = 1, \ldots, N-1) \tag{5.12}$$

i.e. for each $n \neq 0$ we have the algebra of a 'quantum plane' [17, 18]. The well known finite-dimensional representations of the quantum plane (for q a root of unity) lead us again to matrix representations of the differential calculus (see section 5.1).

Using (5.6), the differential of a function f is given by

$$\mathrm{d}f = \sum_{m=0}^{N-1} \mathrm{d}(y^m)\, f_m\,. \tag{5.13}$$

It may appear strange that the differentials $\mathrm{d}y, \mathrm{d}y^2, \ldots$ are linearly independent although y^k depends algebraically on y. Indeed, we shall see in section 6.1 that an additional condition can be imposed on the universal differential calculus which 'corrects' this affair.

5.1. On matrix representations of the differential calculus

The finite-dimensional representations of the 'quantum plane' algebra subject to the relation

$$a\,b = q\,b\,a \tag{5.14}$$

(where q is an Nth primitive root of unity) are given up to equivalence by the $(N \times N)$ matrices

$$a = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & q & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & q^{N-1} \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{5.15}$$

which are known to generate the whole algebra $M_N(\mathbb{C})$ of complex $(N \times N)$-matrices [19] (see also [20]). They satisfy

$$a^N = \mathbf{1} = b^N \tag{5.16}$$

† Acting with d on a relation like (5.12) one should expect to obtain additional relations (commutation relations for 1-forms). This is not so in our case. The relations (5.12) are simply consequences of the general setting of differential calculus as given by (2.7)–(2.9).

where $\mathbb{1}$ denotes the $(N \times N)$ unit matrix. In terms of a and b we can now represent the differential calculus by

$$\rho(y) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \tag{5.17}$$

and

$$\rho(\theta_k) = \begin{pmatrix} 0 & \sum_{\ell=0}^{N-1} c_{k\ell}\, a^\ell\, b^k \\ \sum_{\ell=0}^{N-1} c'_{k\ell}\, a^\ell\, b^k & 0 \end{pmatrix} \tag{5.18}$$

where $c_{k\ell}$ and $c'_{k\ell}$ are complex numbers.

The $*$-involution acts on matrices by Hermitian conjugation. Equation (4.10), i.e. $\theta_k^* = -\theta_{N-k}$, then leads to the condition

$$c_{k\ell}^* = -q^{k\ell}\, c'_{N-k,N-\ell} \tag{5.19}$$

for the constants in (5.18).

The $\star$-involution acts on a matrix B such that $B^\star = P\,B^*\,P$ where the matrix P has entries $P_{ij} = \delta_{i,N-j}$. Then $a^\star = a$ as required by (5.8) and also $b^\star = b$. From $\theta_k^\star = -\theta_k$ we now get the condition

$$c_{k\ell}^* = -q^{k\ell}\, c'_{k\ell} \tag{5.20}$$

(where $*$ denotes complex conjugation).

More general matrix representations of the differential calculus are given by

$$\rho(y) = \begin{pmatrix} \mathbb{1}_M \otimes a & 0 \\ 0 & \mathbb{1}_M \otimes a \end{pmatrix} \qquad \rho(\theta_k) = \begin{pmatrix} 0 & \sum_{\ell=0}^{N-1} c_{k\ell} \otimes a^\ell\, b^k \\ \sum_{\ell=0}^{N-1} c'_{k\ell} \otimes a^\ell\, b^k & 0 \end{pmatrix} \tag{5.21}$$

where the $c_{k\ell}$ and $c'_{k\ell}$ are now $(M \times M)$ matrices. In this case $*$ in (5.19) and (5.20) has to be understood as Hermitian conjugation.

5.2. *Gauge theory on $\mathbb{Z}_2$*

The simplest non-trivial example of a discrete space is a two-point space, of course. This can be endowed with the group structure of $\mathbb{Z}_2$. In this case we have $q = -1$, $y^2 = 1$ and

$$y\,\mathrm{d}y = -\mathrm{d}y\,y\,. \tag{5.22}$$

The two involutions introduced in section 2 coincide in the case under consideration. The field strength of an anti-Hermitian connection A takes the form

$$F = \theta_1\theta_1\,[(R_1 P_1)\,P_1 - 1] = (\theta_1)^2\,[P_1^\dagger P_1 - 1] \tag{5.23}$$

where (4.14) and (4.21) have been used. The 2-form $(\theta_1)^2 = -\frac{1}{4}(\mathrm{d}y)^2$ commutes with all $f \in \mathcal{A}$. As a consequence, the transformation law $F' = UFU^{-1}$ is shared by the coefficient function of F. We can therefore build a gauge-invariant Lagrangian,

$$\mathcal{L} := \mathrm{Tr}(F^\dagger F) = (\theta_1)^4\,\mathrm{Tr}[P_1^\dagger P_1 - 1]^2 \tag{5.24}$$

where Tr denotes the ordinary matrix trace. In order to construct an action, we need a kind of integral, a trace tr acting on forms. Such an integral should only have non-zero values on forms which commute with all functions $f \in \mathcal{A}$. Using the properties which tr has in a representation (see appendix A) one finds

$$S := \mathrm{tr}\mathcal{L} = \mathrm{tr}(\theta_1^4)\; 2\,\mathrm{Tr}[\Phi^\dagger\Phi - 1]^2 \tag{5.25}$$

where we have set $\Phi := P_1(0)$ and used $P_1(1) = \Phi^\dagger$ which follows from (4.21). The constant $\mathrm{tr}(\theta_1^4)$ plays the role of a coupling constant. Equation (5.25) shows that the Yang–Mills action on a two-point space is nothing but the usual Higgs potential, a crucial observation made in [7]. It becomes a field on a manifold $\mathcal{M}$ when the formalism is extended to $\mathcal{M} \times \mathbb{Z}_2$ (see also [23]).

6. Reductions of the universal differential calculus

So far we have dealt with the 'universal' differential calculus on $\mathcal{A}$, i.e. we have only used the general rules (2.7)–(2.9) of differential calculus. There is some freedom to impose additional conditions which are consistent with the universal differential calculus.

When the differential calculus is formulated in terms of the cyclic function y introduced in section 5, there is a natural choice for such a condition in the form of a commutation relation between y and its differential. This is elaborated in section 6.1 and generalized in section 6.2.

Section 6.3 contains a general discussion of reductions of the universal differential calculus which in particular shows that additional relations can be imposed on the differential calculus without changing its dimensionality (i.e. the number of linearly-independent differentials).

6.1. From the universal differential calculus to a calculus on a lattice

The relation

$$y\,\mathrm{d}y = q\,\mathrm{d}y\,y\,. \tag{6.1}$$

leads to a consistent differential calculus on the algebra generated by y [21]. For $N > 2$ this condition is not consistent with the $*$-involution (for which $y^* = y^{-1}$). It is consistent, however, if we choose the $\star$-involution for which $y^\star = y$. From (6.1) we deduce

$$\mathrm{d}y^n = \sum_{m=0}^{n-1} y^m\,\mathrm{d}y\,y^{n-m-1} = [n]_q\,\mathrm{d}y\,y^{n-1} \tag{6.2}$$

where

$$[n]_q := \frac{1-q^n}{1-q}\,. \tag{6.3}$$

In particular,

$$\mathrm{d}y^N = [N]_q\,\mathrm{d}y\,y^{N-1} = 0 \tag{6.4}$$

(since $q^N = 1$) in accordance with $y^N = 1$. This suffices to conclude that (6.1) gives *a consistent reduction of our universal differential calculus. Equation (5.9) is indeed* identically satisfied.

The differential of a function f is now given by

$$\mathrm{d}f = \mathrm{d}y\,\frac{f(qy) - f(y)}{(q-1)\,y} \tag{6.5}$$

which involves the so-called q derivative, and the 1-form θ_0 takes the form†

$$\theta_0 = \frac{1}{1-q}\,\mathrm{d}y\,y^{-1}\,. \tag{6.6}$$

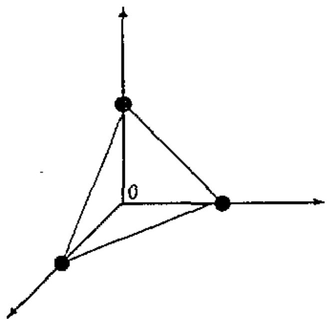

Figure 1. The geometric structure induced on the $N = 3$ point set by the universal differential calculus.

Applying d to (6.1) leads to $(\mathrm{d}y)^2 = 0$.

Remark. For the ordinary differential calculus where $y\,\mathrm{d}y = \mathrm{d}y\,y$ we have $\mathrm{d}y^N = N\,y^{N-1}\mathrm{d}y$ which is *not* consistent with $y^N = \mathbf{1}$ unless $\mathrm{d}y = 0$. Commutative algebras are therefore not in general compatible with the ordinary differential calculus.

In [11, 12] we have considered a certain deformation of the ordinary differential calculus on a manifold. In one dimension, the deformation can be expressed in the form

$$[X, \mathrm{d}X] = \mathrm{d}X\,a \tag{6.7}$$

where X is a coordinate function on $\mathbb{R}$ and a is a positive real constant. An action for a (classical) field theory can be formulated in terms of the deformed differential calculus and turns out to describe a corresponding lattice theory where a plays the role of the lattice spacing. In terms of the new coordinate $y = q^{X/a}$ with $q \in \mathbb{C}$, $q \neq 1$, the commutation relation (6.7) is transformed into (6.1) (see [11] for details). If q is an Nth root of unity, we are considering a closed (periodic) lattice of N points instead of a lattice on the real line.

A differential calculus with M linearly independent differentials on a set G of order N should be thought of as associating M dimensions with it. In the case of the universal differential calculus, the differential $\mathrm{d}f$ of a function f on G involves—as 'partial derivatives'—the differences of the values of f at pairs of points according to (2.13)†. In this sense this differential calculus gives the structure of an $(N-1)$-dimensional polyhedron in N dimensions to the set G (where the N points of G appear as the vertices).

One also arrives at such a picture in the following way. Let x_n be coordinate functions on $\mathbb{R}^N$. We may then consider the equations (2.3) and (2.4) as algebraic equations imposed on the functions x_n. Their solutions determine a set of N points in $\mathbb{R}^N$ which form the vertices of an $(N-1)$-dimensional polyhedron (see figure 1).

It is hard to see how one should formulate and understand the reduction of the differential calculus in terms of the 'coordinates' x_n (cf section 6.3, however). The reformulation in terms of the single function y makes it easy to formulate a constraint, i.e., (6.1), which reduces the $N-1$ dimensions of the universal differential calculus to a single one. The corresponding geometric picture (based on (6.5)) is obtained by drawing the set of Nth roots of unity in the (complex) plane (see figure 2).

† *If we consider the differential calculus with (6.1) on the algebra of functions on* $\mathbb{R}$ *where* q *is not a root of* unity, the 1-form on the right-hand side plays a special role as a measure for integrating functions of y. The corresponding integral sums the values of the function on a q-lattice (cf [11, equation (4.20)]).

† (2.13) actually only involves the differences $f(g) - f(e)$ which suggests only drawing lines from e to the other points of the set. Note, however, that the choice of e is arbitrary.

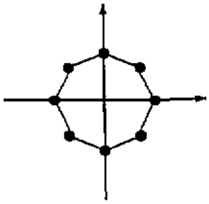

Figure 2. The $N = 8$ point set as a one-dimensional closed lattice embedded in the two-dimensional plane (the structure given to it by the one-dimensional differential calculus).

6.2. $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ reductions

In section 5 we considered the group structure $\mathbb{Z}_N$ on a set of N elements. The formulae given there are easily generalized, using multi-index notation, to a group structure $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_r}$ where $N = N_1 \cdots N_r$. Let $\underline{n} \in G$ and

$$y^{\underline{n}} := \prod_{k=1}^{r} y_k^{n_k} \qquad y_k^{N_k} = \mathbf{1} \tag{6.8}$$

$$q^{\underline{n}} := \prod_{k=1}^{r} q_k^{n_k} \qquad q_k^{N_k} = 1 \tag{6.9}$$

(q_k primitive roots). Then

$$y^{\underline{m}}\,\mathrm{d}y^{\underline{n}} + \mathrm{d}y^{\underline{m}}\,y^{\underline{n}} = \mathrm{d}y^{\underline{m}+\underline{n}}\,. \tag{6.10}$$

Generalizing the differential calculus reduction scheme of section 6.1, we impose the commutation relations

$$y_k\,\mathrm{d}y_\ell = q_k^{\delta_{k\ell}}\,\mathrm{d}y_\ell\,y_k\,. \tag{6.11}$$

As a consequence, we have

$$\mathrm{d}y^{\underline{n}} = \sum_{k=1}^{r} [n_k]_{q_k}\,\mathrm{d}y_k\,y^{\underline{n}-\underline{e}_k} \tag{6.12}$$

where $\underline{e}_k \in G$ has components $e_{k\ell} = \delta_{k\ell}$. Furthermore,

$$y^{\underline{m}}\,\mathrm{d}y^{\underline{n}} = \sum_{k=1}^{r} [n_k]_{q_k}\,q_k^{m_k}\,\mathrm{d}y_k\,y^{\underline{m}+\underline{n}-\underline{e}_k}\,. \tag{6.13}$$

Equation (6.11) has to be compatible with (6.10). Using the last two equations in (6.10), we obtain

$$0 = \sum_{k=1}^{r} \left([n_k]_{q_k}\,q^{m_k} + [m_k]_{q_k} - [m_k + n_k]_{q_k}\right)\mathrm{d}y_k\,y^{\underline{m}+\underline{n}-\underline{e}_k} \tag{6.14}$$

which is identically satisfied since the expression in round brackets on the right-hand side vanishes identically. This shows that (6.11) indeed defines a consistent reduction of the universal differential calculus.

The geometric structure which the reduced differential calculus places on the N-point set is a 'discrete torus'. It is a cartesian product of 'discrete circles' like the one shown in figure 2.

Example: $\mathbb{Z}_2 \times \mathbb{Z}_3$

The constraints

$$u^2 = \mathbf{1} \qquad v^3 = \mathbf{1} \tag{6.15}$$

imposed on the two coordinates u, v on $\mathbb{R}^2$ determine a set of six points. In terms of u and v, the universal differential calculus on the six-point space is given by the following set of rules:

$$\begin{aligned}
&u\,\mathrm{d}u + \mathrm{d}u\,u = 0 && v\,\mathrm{d}u + \mathrm{d}v\,u = \mathrm{d}(uv) \\
&u\,\mathrm{d}v + \mathrm{d}u\,v = \mathrm{d}(uv) && v\,\mathrm{d}v + \mathrm{d}v\,v = \mathrm{d}(v^2) \\
&u\,\mathrm{d}(uv) + \mathrm{d}u\,uv = \mathrm{d}v && v\,\mathrm{d}(uv) + \mathrm{d}v\,uv = \mathrm{d}(uv^2) \\
&u\,\mathrm{d}(v^2) + \mathrm{d}u\,v^2 = \mathrm{d}(uv^2) && v\,\mathrm{d}(v^2) + \mathrm{d}v\,v^2 = 0 \\
&u\,\mathrm{d}(uv^2) + \mathrm{d}u\,uv^2 = \mathrm{d}(v^2) && v\,\mathrm{d}(uv^2) + \mathrm{d}v\,uv^2 = \mathrm{d}u\,.
\end{aligned} \tag{6.16}$$

The 1-form θ_0 takes the form

$$\theta_0 = \tfrac{1}{6}[\mathrm{d}u\,u + \mathrm{d}v\,v^2 + \mathrm{d}(uv)\,uv^2 + \mathrm{d}(v^2)\,v + \mathrm{d}(uv^2)\,uv]\,. \tag{6.17}$$

Any function f on the set can be written as

$$f = \sum_{i=0}^{1}\sum_{j=0}^{2} u^i\,v^j\,f_{ij} \tag{6.18}$$

with constants f_{ij}. This leads to the expression

$$\mathrm{d}f = \sum_{i=0}^{1}\sum_{j=0}^{2} \mathrm{d}(u^i v^j)\,f_{ij}\,. \tag{6.19}$$

for its differential. The reduction is now performed by imposing the relations

$$v\,\mathrm{d}v = p\,\mathrm{d}v\,v \qquad v\,\mathrm{d}u = \mathrm{d}u\,v \qquad u\,\mathrm{d}v = \mathrm{d}v\,u \tag{6.20}$$

where p is a cubic primitive root of unity. They imply

$$\mathrm{d}(uv) = \mathrm{d}u\,v + \mathrm{d}v\,u \tag{6.21}$$

$$\mathrm{d}(v^2) = (1+p)\,\mathrm{d}v\,v \tag{6.22}$$

$$\mathrm{d}(uv^2) = \mathrm{d}u\,v^2 + (1+p)\,\mathrm{d}v\,u\,v\,. \tag{6.23}$$

Using these relations in (6.19) one finds

$$\mathrm{d}f = \mathrm{d}u\,\frac{f(u,v) - f(-u,v)}{2u} + \mathrm{d}v\,\frac{f(u,pv) - f(u,v)}{(p-1)v}\,. \tag{6.24}$$

Furthermore,

$$\theta_0 = \frac{1}{2}\,\mathrm{d}u\,u^{-1} - \frac{1}{p-1}\,\mathrm{d}v\,v^{-1} \tag{6.25}$$

and we obtain the 2-form relations

$$\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v\,\mathrm{d}u = 0 \qquad \mathrm{d}v\,\mathrm{d}v = 0\,. \tag{6.26}$$

In the sense of our discussion in section 6.1, via (6.24) the reduced differential calculus gives a two-dimensional torus structure to the set of six points.

6.3. Further remarks about reductions

So far we have understood a *reduction* of a differential calculus as a procedure to reduce its dimensionality (i.e. the number of linearly independent differentials). This is done by adding extra relations to the differential calculus. It is, however, possible to add relations without changing the dimension.

In section 2 we have introduced the 1-forms θ_{ij} and shown that setting one (or several) of these forms to zero does not lead to any further constraint on the first-order differential calculus. It leads, of course, to 2-form relations $d\theta_{ij} = 0$ and additional conditions according to the involution which is used.

With a differential calculus we can associate a directed graph with N vertices. An arrow connects vertex i with vertex j whenever $\theta_{ij} \neq 0$. For the universal differential calculus this means that each pair of vertices is connected by two lines with opposite direction. We can represent this graph by the $(N \times N)$-matrix which has zeros on the diagonal and all other entries equal to 1. A 1 in the ith row and jth column stands for an arrow pointing from vertex i to vertex j. Setting $\theta_{12} = 0$, for example, means that we have to delete the arrow from vertex 1 to vertex 2 (in the corresponding matrix, the 1 in the first row and second column has to be replaced by 0). Imposing also $\theta_{21} = 0$ separates the two vertices (which may, however, still be connected via other vertices).

Example. In section 6.2 a reduction of the universal differential calculus on the six-point space to a two-dimensional differential calculus was considered. Using

$$u = x_0 - x_1 + x_2 - x_3 + x_4 - x_5 \tag{6.27}$$

$$v = x_0 + p^2 x_1 + p\, x_2 + x_3 + p^2 x_4 + p\, x_5 \tag{6.28}$$

(where p is a primitive cubic root of unity), the reduction conditions (6.20) can be expressed in the form $\theta_{ij} = 0$ for certain values of the indices i, j. These equations are summarized in the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{6.29}$$

The corresponding graph is shown in figure 3. That each row and each column of the matrix have precisely two 1s is related to the fact that the reduced differential calculus is two-dimensional. For the graph it means that each vertex has two incoming and two outgoing arrows. In general, the reduction procedure leads to graphs where the number of incoming and outgoing arrows varies from vertex to vertex. This somehow means that the dimension varies over the set of points.

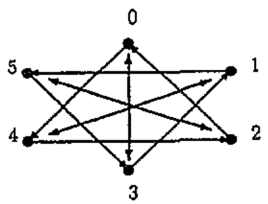

Figure 3. The graph associated with a two-dimensional reduction of the universal differential calculus on a six-point space.

7. Conclusions

We have formulated differential calculus and gauge theory on an arbitrary finite set. Endowing the latter with a group structure, one can define analogues of Maurer–Cartan forms (see also [9]). It is then straightforward to define principal fibre bundles with discrete structure groups and connections on them. Other differential geometric structures are expected to also have a 'discrete analogue'.

In the special example of gauge theory on a two-point space we recovered the geometric interpretation of the Higgs field as in Connes' formulation of the standard model ([7], see also [23]).

The universal differential calculus (in the sense of Connes) associates with all linearly *independent elements of an algebra corresponding linearly independent differentials*. This means that it assigns a geometric picture (a polyhedron) in N dimensions to a set of N elements. The universal differential calculus admits reductions to consistent differential calculi with which one can associate a similar geometric picture in lower dimensions. We have only given examples which certainly do not exhaust the possibilities. In particular, we discussed a reduction to a single dimension in section 6.1. In [11] the resulting differential calculus has been shown to be equivalent to a 'non-commutative' differential calculus on an equidistant periodic lattice.

There is one point which we would like to stress here. In an algebraic sense, one can easily construct reductions of the universal differential calculus. The problem is that, in general, one is not able to find some geometric picture associated with such a reduction. Such a picture may be found by expressing the reduced calculus in suitable 'coordinates'. However, we do not yet have a systematic way to find such coordinates. On the other hand, we have shown that associated with certain group structures there are choices of coordinates in terms of which reductions of the universal differential calculus lead to a geometric understanding of the resulting differential calculus.

It is important, however, to keep the following in mind. When we speak about a '*geometric picture*' *we are actually guided by continuum geometry, thinking of spheres*, tori etc. A finite set of points can be connected in such a way that the resulting structure reminds us of such a continuum geometric picture. But there are other connection structures for which no corresponding continuum picture exists. This means that there are many more possibilities for discrete structures. This also concerns the dimensionality. To a set of points we can assign different dimensions. The fact that for a discrete set there is no rigid notion of dimension has some interesting aspects. If a spacetime model is set up in such a framework, the dimension may even change with length scale and in such a way incorporate features of Kaluza–Klein theories (cf [6]).

In [11, 12] we were interested in deformations of the ordinary differential calculus on the algebra of functions on $\mathbb{R}^N$. In the case of a certain deformation (cf (6.7)) it turned out that the differential calculus could be restricted to functions on a lattice. In the same way we can understand each of the differential calculi of the present paper as a calculus on $\mathbb{R}^M$ when the calculus has M independent differentials. The calculus still contains the information about the point set in the following way. Consider, for example, the differential calculus with $u\,\mathrm{d}u = -\mathrm{d}u\,u$ where u is a real coordinate on $\mathbb{R}$. Using the Leibniz rule, it implies $\mathrm{d}(u^2) = 0$ which means that u^2 is a constant with respect to the differential calculus under consideration. It follows that the 'constants' are precisely the even functions h of u (i.e. $h(-u) = h(u)$). Since every function f can be written in a unique way as $f = h_0 + h_1 u$ with even functions h_0, h_1, it is represented by a pair of 'constants' and in this sense we have a two-point space.

As already mentioned in the introduction, there are several approaches towards physical theories based on discrete spaces in the literature. Non-commutative differential geometry of discrete spaces should have some impact on these approaches and vice versa.

Acknowledgments

AD is grateful to the Heraeus-Foundation for financial support during a visit at the Institute for Theoretical Physics in Göttingen. We are also grateful to John Madore for pointing out an error in the first version of this paper.

Appendix A. A representation of the differential calculus on $\mathbb{Z}_2$

Following Connes' treatment of the two-point space [7], we represent a function f on $\mathbb{Z}_2$ by a diagonal matrix

$$f = \begin{pmatrix} f(0)\,\mathbf{1} & 0 \\ 0 & f(1)\,\mathbf{1} \end{pmatrix} \tag{A.1}$$

where $\mathbf{1}$ denotes the $(m \times m)$ unit matrix. The differential of f is represented by

$$\mathrm{d}f = [f(0) - f(1)]\mathrm{i} \begin{pmatrix} 0 & -M^\dagger \\ M & 0 \end{pmatrix} \tag{A.2}$$

with a complex $(m \times m)$ matrix M. In particular, the function y introduced in section 5 is given by

$$y = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{A.3}$$

and we find

$$\mathrm{d}y = 2\,\mathrm{i} \begin{pmatrix} 0 & -M^\dagger \\ M & 0 \end{pmatrix} \qquad \mathrm{d}y^2 = 0\,. \tag{A.4}$$

For the 1-form θ_1 we obtain

$$\theta_1 = -\tfrac{1}{2}\,\mathrm{d}y\,y = -\mathrm{i} \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} \tag{A.5}$$

so that

$$(\theta_1)^4 = \begin{pmatrix} (M^\dagger M)^2 & 0 \\ 0 & (M M^\dagger)^2 \end{pmatrix} \tag{A.6}$$

and therefore

$$\mathrm{tr}[(\theta_1)^4 f] = 2\,\mathrm{tr}(M^\dagger M)^2\,[f(0) + f(1)] \tag{A.7}$$

where tr is the ordinary matrix trace.

Appendix B. Matrix algebras as differential algebras

Let M_N denote the algebra of complex $(N \times N)$ matrices and Ω the direct sum $M_N \oplus M_N$. The latter becomes an algebra with the multiplication rule $(A, B)(A', B') = (AA', BB')$. There is a natural $\mathbb{Z}_2$ grading. We call an element of Ω 'even' if it is of the form (A, A) and 'odd' if it has the form $(A, -A)$. Since $(A, B) = ((A + B)/2, (A + B)/2) + ((A - B)/2, -(A - B)/2)$, Ω splits into a direct sum, $\Omega = \Omega^+ \oplus \Omega^-$. Let $\mathcal{A}_N$ denote the commutative subalgebra of Ω^+ consisting of elements of the form $f = (F, F)$ where $F \in M_N$ is diagonal. In terms of the matrices E_{ij} with components $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$ we have

$$f = \sum_{i=0}^{N-1} f_i\,(E_{ii}, E_{ii})\,. \tag{B.1}$$

For $i \neq j$ we introduce $\theta_{ij} := c_{ij}\,(E_{ij}, -E_{ij})$ where $c_{ij} \in \mathbb{C}$. Furthermore, we define

$$\mathrm{d}f := [f, \vartheta] \qquad \vartheta := \sum_{i\neq j} \theta_{ij} \tag{B.2}$$

and $\mathrm{d}(f_0\mathrm{d}f_1 \cdots \mathrm{d}f_r) := \mathrm{d}f_0\,\mathrm{d}f_1 \cdots \mathrm{d}f_r$. With these definitions, Ω has the structure of a differential calclulus over $\mathcal{A}_N$.

For $x_i := (E_{ii}, E_{ii})$ we find $\sum_{i=0}^{N-1} \mathrm{d}x_i = 0$ and $\mathrm{d}x_i\,x_j = \theta_{ij}$ for $i \neq j$ (cf section 2). There are $N-1$ independent differentials $\mathrm{d}x_i,\ i = 1, \ldots, N-1$ in this calculus. Setting some of the c_{ij} to zero, it is possible to reduce the number of independent differentials. In this way one can turn M_N itself into a differential algebra. As an example, let us consider M_3. Let $\mathcal{A}_3$ be the subalgebra of diagonal matrices and

$$\vartheta := \begin{pmatrix} 0 & 0 & c_{02} \\ 0 & 0 & c_{12} \\ c_{20} & c_{21} & 0 \end{pmatrix} \tag{B.3}$$

(i.e. we set $c_{01} = c_{10} = 0$). With the above definition of d, M_3 becomes a differential algebra Ω_3 over $\mathcal{A}_3$. In this case, we have a $\mathbb{Z}_2$ grading $\Omega_3 = \Omega^+ \oplus \Omega^-$ where Ω^+ and Ω^- consist, respectively, of the matrices of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \tag{B.4}$$

with possible non-zero entries indicated by a $*$.

References

[1] Ambarzumian V and Iwanenko D 1930 Zur Frage nach Vermeidung der unendlichen Selbstrückwirkung des Elektrons *Z. Phys.* **64** 563
Ruark A 1931 The roles of discrete and continuous theories in physics *Phys. Rev.* **37** 315
Snyder H S 1947 Quantized spacetime *Phys. Rev.* **71** 38

[2] Finkelstein D 1969 Space-time code *Phys. Rev.* **184** 1261

[3] Yamamoto H 1984 Quantum field theory on discrete spacetime *Phys. Rev.* **30** 1727; 1985 Quantum field theory on discrete spacetime. II *Phys. Rev.* **32** 2659
Yamamoto H, Hayashi A, Hashimoto T and Horibe M 1993 Towards a canonical formalism of field theory on discrete spacetime *Preprint* (Fukui, Japan)

[4] 't Hooft G 1990 Quantization of discrete deterministic theories by Hilbert space extension *Nucl. Phys.* B **342** 471

[5] Feynman R P 1982 Simulating physics with computers *Int. J. Theor. Phys.* **21** 467
Minsky M 1982 Cellular vacuum *Int. J. Theor. Phys.* **21** 537

[6] Bombelli L, Lee J, Meyer D and Sorkin R D 1987 Space-time as a causal set *Phys. Rev. Lett.* **59** 521

[7] Connes A 1990 Essay on physics and non-commutative geometry *The Interface of Mathematics and Particle Physics* ed D Quillen, G Segal and S Tsou (Oxford: Oxford University Press) p 9
Connes A and Lott J 1990 Particle models and non-commutative geometry *Nucl. Phys. (Proc. Suppl.)* B **18** 29; 1992 The metric aspect of non-commutative geometry *New Symmetry Principles in Quantum Field Theory* ed J Fröhlich *et al* (New York: Plenum)

[8] Coquereaux R, Esposito-Farese G and Vaillant G 1991 Higgs fields as Yang-Mills fields and discrete symmetries *Nucl. Phys.* B **353** 689

Kastler D 1991 A detailed account of Alain Connes' version of the standard model in non-commutative geometry *Marseille preprints* CPT-91/P.2610, CPT-92/P.2894

Chamseddine A H, Felder G and Fröhlich J 1992 Unified gauge theories in non-commutative geometry *Phys. Lett.* **296B** 109

Varilly J C and Gracia-Bondia J M 1992 Connes' non-commutative differential geometry and the standard model *Zaragoza preprint* DFTUZ/92/20

[9] Sitarz A 1992 Noncommutative geometry and gauge theory on discrete groups *Preprint* TPJU-7/92; 1992 Non-commutative geometry and the Ising model *Preprint* TPJU-18/92

[10] Connes A 1986 Non-commutative differential geometry *Publ. IHES* **62** 257

[11] Dimakis A and Müller-Hoissen F 1992 Quantum mechanics on a lattice and q-deformations *Phys. Lett.* **295B** 242

[12] Dimakis A, Müller-Hoissen F and Striker T 1993 From continuum to lattice theory via deformation of the differential calculus *Phys. Lett.* **300B** 141; 1993 Noncommutative differential calculus and lattice gauge theory *J. Phys. A: Math. Gen.* **26** 1927

[13] Dimakis A and Müller-Hoissen F 1993 Stochastic differential calculus, the Moyal $*$-product, and non-commutative geometry *Lett. Math. Phys.* **28** 123

[14] Fanchi J R 1993 Review of invariant time formulations of relativistic quantum theories *Found. Phys.* **23** 487

[15] Dimakis A and Müller-Hoissen F 1992 Noncommutative differential calculus, gauge theory and gravitation *Göttingen preprint* GOE-TP 33/92

[16] Coquereaux R 1989 Noncommutative geometry and theoretical physics *J. Geom. Phys.* **6** 425

[17] Manin Yu 1988 Quantum groups and non-commutative geometry *Montreal preprint* CRM-1561

[18] Wess J and Zumino B 1990 Covariant differential calculus on the quantum hyperplane *Nucl. Phys. (Proc. Suppl.)* B **18** 302

[19] Morris A O 1967 On a generalized Clifford algebra *Quart. J. Math. Oxford* **18** 7

[20] Floratos E G 1989 Representations of the quantum group $GL_q(2)$ for values of q on the unit circle *Phys. Lett.* **233B** 395

Weyers J 1990 The quantum group $GL_q(n)$ and Weyl-Heisenberg operators *Phys. Lett.* **240B** 396

[21] Manin Yu I 1991 Notes on quantum groups and quantum de Rham complexes *Bonn preprint* MPI/91-60

[22] Hamermesh M 1962 *Group Theory and its Application to Physical Problems* (Oxford: Pergamon)

[23] Sitarz A 1993 Higgs mass and non-commutative geometry *Phys. Lett.* **308B** 311